International Journal of Theoretical Physics, Vol. 32, No. 2, 1993

Pre-Regge Calculus: Topology via Logic

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Received June 2, 1992

Spacetime is simulated by a pattern space—a finite topological space homotopically equivalent to the spacetime simulated. Unlike the Regge calculus, we ignore metrical properties. A calculus based on Boolean arithmetic is suggested to describe the changing of global topology of the pattern spaces, and is suitable for computer realization.

1. INTRODUCTION

Within the framework of Regge calculus (Regge, 1961), a real spacetime is replaced by a polyhedron which is a pattern space. This means that, given a polyhedron considered as a pattern space, the spacetime can be restored up to its *metrical* properties. Thus, the Regge calculus serves to describe a spacetime that is already defined somehow. However, it scarcely describes the dynamics of a global topology.

The idea of using logic as a ground on which the topology grows belongs to Wheeler (Misner *et al.*, 1973), who proposed to consider events *per se*, not being placed into a spacetime. The calculus I propose describes the construction of spacetime from events.

The spacetime is replaced (or simulated) by a finite topological space, called *pattern space.* The pattern space restores merely the topology of the spacetime, saying nothing about its geometry. That is why this calculation scheme is called *pre*-Regge calculus—it is the step preceding the building of Regge space (Regge, 1961).

The pattern space is represented as a directed graph (digraph). The vertices of the digraph are spacetime points (or elementary events), while

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the arrows linking the vertices show their tendency to each other. The coarser the topology is, the more arrows the pattern space contains.

The apparent mathematical tool to describe the dynamics of the topology of pattern spaces is the *Boolean machinery* (Section 3). Each pattern space is associated with a Boolean matrix and the transition from one topology to another is described in terms of Boolean arithmetics.

The construction of new pattern spaces by given standard constituents is performed by means of *surgery* (Section 4). Sawing and pasting are described as weakening of the entire topology, while cutting strengthens the topology. All are described via two elementary operations with pattern spaces: *stretching* and *cutting* (which can be described in turn in Boolean language). In some sense the proposed calculus is the realization of Wheeler's spacetime machine (Misner *et al.*, 1973) (Figure 1).

Section 5 considers the description of the dynamics of the global topology within the bounds of pre-Regge calculus. The *Boolean superspace*—the space of pattern spaces—is introduced as the collection of all finite topological spaces together with an infinitely large number of separated "spare" points. The transition from one global topology to another is described as the variety of all possible paths in the Boolean superspace. Each path, in turn, can be decomposed into a sequence of elementary transitions, each of which is the action of a stretching or cutting operator. Consequently, if there is an amplitude (or something like an amplitude) assigned to each elementary transition, the Feynman sum for the transition amplitude between any two pattern spaces can be used.

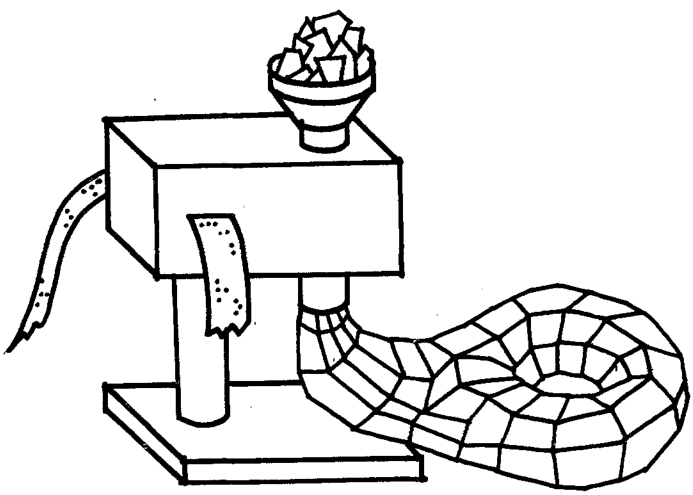

Fig. 1. The Wheeler spacetime machine.

2. QUASIORDERS AND GRAPHS ASSOCIATED WITH FINITE TOPOLOGIES

In this section the mathematical structure of pattern spaces will be described in detail. Let X be a *finite topological* (FT) *space*. That is, X is finite and a collection τ of subsets of X, called *open* subsets, is such that

$$\begin{gathered}\emptyset, X \in \tau \\ A, B \in \tau \quad \text{implies} \quad A \cap B \in \tau \\ A_i \in \tau, \quad i \in I \quad \text{implies} \quad \bigcup \{A_i \mid i \in I\} \in \tau\end{gathered} \tag{2.1}$$

The elements of X will be called points. Let $x \in X$ be a point. A neighborhood of x is any open subset of X containing x. Since X is finite, the intersection of all neighborhoods of a point x is an open set; call it the open monad of x and denote it $(x)_\tau$ or simply (x) if no ambiguity occurs:

$$(x)_\tau := \bigcap \{A \mid x \in A \in \tau\} \tag{2.2}$$

The collection $\{(x) \mid x \in X\}$ is the base of the topology τ, namely each subset of X is open in τ if and only if it is the union of elements of the base:

$$A \in \tau \quad \text{if and only if} \quad A = \bigcup \{(a)_\tau \mid a \in A\}$$

or, in other words, *any open set contains the open monad of each of its points.* Now define the binary relation, denote it also τ, on X as

$$x \tau y \quad \text{if and only if} \quad y \in (x)_\tau \tag{2.3}$$

The defined relation τ possesses the properties of reflexivity ($x \tau x$ for any x) and transitivity ($x \tau y$ and $y \tau z$ implies $x \tau z$), hence it is a quasiorder.

The relation (2.3) is called the *quasiorder* associated with the finite topological space (X, τ).

Moreover, given a quasiorder τ on X, define for each point its upper cone $u\tau(x)$:

$$u\tau(x) := \{y \in X \mid x \tau y\}$$

The collection of all upper cones $\{u\tau(x) \mid x \in X\}$ forms the base of a topology on X called the topology generated by the quasiorder τ, so that the following theorem holds:

Given a finite set X, the topologies and quasiorders on X are in one-to-one correspondence. Given a topology τ, the quasiorder associated with τ generates just the topology τ, and vice versa.

Each finite quasiordered space (X, τ) can be associated with a directed graph (digraph) $G(X, \tau)$, called the *Hasse diagram* of (X, τ), in the following way. The set of vertices of $G=G(X, \tau)$ is the set X_1 and the arrows of G connect quasiordered pairs. This graph is the mathematical realization of pattern space [the graph definitions and notations are borrowed from Tutte (1984)].

The *Hasse graph* of a finite topological space (X, τ) is the Hasse diagram of the quasiorder associated with the topology τ.

The Hasse graph of an FT space is always reflexive (each vertex is connected with itself) and transitive (if there are arrows from x to y and from y to z, then there is an arrow from x to z). Thus no ambiguity occurs if in drawing the graph one always omits loops (arrows from x to x) and sometimes omits composite edges. Consider some examples. Let $X=\{a, b, c\}$ be a three-element set.

Example 1. $\tau=\{\varnothing, a, \{a, b\}, X\}$. Using Isham's (1989) notation, one can omit braces and it can be written as $\tau=a(ab)$. The monads are $(a)=a$, $(b)=ab$, $(c)=abc$; thus the quasiorder τ is $c\tau b$, $c\tau a$, $b\tau a$, and the Hasse graph is as shown in Figure 2.

Example 2. The discrete topology δ on X is $\delta=2^x=abc$ (any subset is open). The monads are $(a)=a$, $(b)=b$, $(c)=c$, and the quasiorder δ is the equality relation given in Figure 3.

Example 3. The undiscrete topology $o=\{\varnothing, X\}$. In this case $(a)=(b)=(c)=X$ and any pair is quasiordered (Figure 4).

Fig. 2. $\tau=a(ab)$.

Fig. 3. The Hasse graph is totally disjoint.

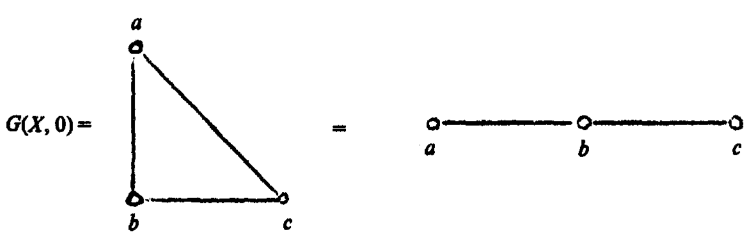

Fig. 4. The Hasse graph is the 3-clique.

A *clique* Q is a complete subdigraph of G, namely each pair of vertices q, $q' \in Q$ are connected by the arrows $q \to q'$ and $q' \to q$. In the case when there are arrows, say, from a to b as well as from b to a, the arrows are omitted.

Now consider the collection $T = T(X)$ of all topologies on X. As is outlined in Isham (1989) and Larson and Andima (1975), the set T is partially ordered by set inclusion (since any topology is a set of sets). Given two topologies α, $\beta \in T$, α is said to be *weaker* (*coarser*) than β and β is *stronger* (*finer*) than α if $\alpha \subset \beta$, or, in other words, any set open in α is open in β. Moreover, T is the lattice (suprema and infima do exist for each pair of topologies) with the greatest element δ (the discrete topology) and the least element o (the undiscrete topology). This partial order can be expressed in terms of Hasse graphs. Let α, $\beta \in T$, α is weaker than β, $\alpha \cap \beta$, and P, Q be the Hasse graphs associated with the topologies α, β, respectively. Evidently, the graphs P and Q have the same sets of vertices. The following statement always holds:

Given two topologies on the set X, the Hasse graph of stronger topology is the subdigraph of the Hasse graph associated with the weaker topology.

To corroborate this fact, consider an arrow of Q: $x\beta y$ means that $y \in (x)_\beta = \bigcap\{A \mid x \in A \in \beta\}$. However, the α-monad $(x)_\alpha$ is the intersection $\bigcap\{A \mid x \in A \in \alpha\}$ of a greater number of sets (since $\alpha \subset \beta$), hence $(x)_\beta \subset (x)_\alpha$; thus $x\beta y$ implies $x\alpha y$ and each arrow of Q is an arrow of P. This reasoning can be elucidated by comparison of Examples 1–3 cited above.

The lattice operations on T, joins (suprema) and meets (infima), can also be described in terms of graphs. Let α and β be two topologies on X associated with the Hasse graphs P and Q, respectively. The join $\alpha \vee \beta$ is the weakest topology which is stronger than both α and β. In terms of Hasse graphs that means that the graph H associated with the topology $\alpha \vee \beta$ must be a subdigraph of both P and Q. In addition, any common subdigraph of both P and Q is a subdigraph of H. Therefore the graph H of the join topology $\alpha \vee \beta$ is the digraph whose arrows are common arrows of P and Q:

$$G(X, \alpha \vee \beta) = G(X, \alpha) \cap G(X, \beta) \tag{2.4}$$

(Evidently, this graph is reflexive and transitive.)

Now consider the graph K associated with the meet $\alpha \wedge \beta$, which must contain all arrows of P as well as all arrows of Q. However, the graph K_0 whose arrows are exhausted by those of P and Q is reflexive, but may not be transitive, hence $K_0 = P \cap Q$ is not a Hasse graph in general. To obtain the required graph K, all possible composite arrows must be added to K_0. This operation is called the transitive closure of the graph K_0. So, the graph

K of the meet topology $\alpha \wedge \beta$ is the transitive closure of the graph whose arrows are the arrows of P or Q:

$$G(X, \alpha \wedge \beta) = \mathbf{TCl}\,(G(X, \alpha) \cup G(X, \beta)) \tag{2.5}$$

The operation **TCl** is adequately described in terms of Boolean matrices.

3. BOOLEAN MACHINERY

The calculus described in this section is based on Boolean arithmetics. It operates with vectors and matrices over the Boolean space $B^1 = \mathscr{B}$, which is the two-element ordered set $\mathscr{B} = \{0, 1\}$ with the operations of conjunction (or product $\wedge$), disjunction (or sum $\vee$), addition modulo 2 (or symmetric difference $\oplus$) and negation ($\neg$) defined for $x, y \in \mathscr{B}$ as follows:

x	y	xy (or $x \wedge y$)	$x \vee y$	$x \oplus y$	$\bar{x}$ (or $\neg x$)	
0	0	0	0	0	1	
0	1	0	1	1	1	(3.1)
1	0	0	1	1	0	
1	1	1	1	0	0	

The n-dimensional Boolean space B^n consists of all n-tuples of elements of $\mathscr{B}$ with the operations (3.1) defined coordinatewise. Boolean spaces are naturally partially ordered: given two vectors $g, h \in B^n$,

$$g \leq h \quad \text{if and only if} \quad g_i \leq h_i \quad \text{for any} \quad i = 1, \ldots, n \tag{3.2}$$

This is called the binomial order on B^n.

Any *labeled* directed graph G of n vertices can be unambiguously associated with a Boolean $n \times n$ matrix, called its incidence matrix G_{ij}, in the following way: the vertices are enumerated by $1, \ldots, n$, and

$$G_{ij} = \begin{cases} 1 & \text{if there is an arrow stretching} \\ & \text{from vertex } i \\ & \text{to vertex } j \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

The Hasse graphs associated with finite topologies have the special features outlined in the previous section: reflexivity and transitivity. In Boolean terms, the reflexivity is expressed as

$$G_{ii} = 1 \qquad \text{for any} \quad i = 1, \ldots, n \tag{3.4}$$

while the transitivity is expressed as

$$G_{ik}=1 \quad \text{and} \quad G_{kj}=1 \quad \text{implies} \quad G_{ij}=1 \tag{3.5}$$

The usual matrix product in the space of all Boolean $n \times n$ matrices can be introduced:

$$(PQ)_{ij}=\bigvee_{k=1}^{a} (P_{ik}Q_{kj}) \tag{3.6}$$

Denote by $T(X)$, or $T(n)$, or simply T if no ambiguity occurs, the set of all Boolean matrices which are incidence matrices of Hasse graphs associated with topologies on a set X, of cardinality n. It follows from the condition (3.4) that

$$(G^a)_{ij}=(GG)_{ij}\geq G_{ij} \tag{3.7}$$

and the transitivity condition (3.5) yields

$$(GG)_{ij}\leq G_{ij}$$

Hence the necessary and sufficient condition for a Boolean matrix G to be an element of T is

$$\begin{aligned} &G_{ii}=1 \\ &G_{ij}=(G^m)_{ij} \qquad \text{for any positive integer } m \end{aligned} \tag{3.8}$$

The operations on topologies can be translated into Boolean language. Let $\alpha, \beta \in T(X)$ be topologies on X, and P, Q be their Hasse graphs and P_{ij}, Q_{ij} be the incidence matrices of P, Q, respectively. Let K be the Hasse graph of the join $\alpha \vee \beta$. Hence its incidence matrix K_{ij}, as follows from (2.4), is

$$K_{ij}=P_{ij}Q_{ij} \tag{3.9}$$

To avoid confusion with tensor notation, I emphasize that in this paper *there is no summation over repeating indices.* All operations will be written explicitly as, say, (3.6).

To describe the meet $\alpha \wedge \beta$, first note that the graph K_0 described at the end of Section 2 has the incidence matrix $K_{0ij}=P_{ij} \vee Q_{ij}$. Its transitive closure $K=\mathrm{TCl}\,K_0$ is the least [under the partial order (3.2)] matrix satisfying the conditions (3.8). So K_{ij} has the general form

$$K_{ij}=\bigvee_{m=1}^{\infty} (K_0^m)_{ij}$$

Consequently, due to (3.4) and (3.7), it is equivalent to

$$K=(K_0)^{n-1}=(P \vee Q)^{n-1} \tag{3.10}$$

where n is the cardinality of the set X.

The Boolean machinery I suggest consists of tools which allows one to describe transitions from a topology α to some other one β on a given set X. This transition will be described as a travel along the lattice $T=T(X)$ of all topologies of X. Each step of this travel will be an elementary operation of minimal change of topology: weakening or strengthening it. Two types of such elementary operations will be described. The first is the minimal weakening of topology, it will be called stretching (an arrow). The second strengthens the topology and I shall call it a cutting (or separation of cliques). Let us consider them in detail.

The stretching operator $\mathbf{S}^{pq}$ $(p, q \in X, p \neq q)$ weakens a given topology α in such a way that $\mathbf{S}^{pq}\alpha$ is (i) the strongest topology which is not stronger than α and (ii) whose Hasse graph contains the arrow stretched from the vertex p to q. First consider this operation in terms of Hasse graphs. Let G be the Hasse graph associated with α. Evidently, if G already contains the arrow $p \to q$ the operator $\mathbf{S}^{pq}$ does not change the topology. Suppose G does not have this arrow. Adding it to G, a new graph, denote it $G \cup \Delta^{pq}$, is obtained which is not transitive in general. To make it transitive one must also stretch the arrows from each vertex connected with p (including p itself) to each vertex with which q is connected (including q itself). Consider the example of Figure 5. In terms of Boolean matrices the stretching operator looks like

$$(\mathbf{S}^{pq}G)_{ij} = G_{ij} \vee (G_{ip} G_{qj}) \tag{3.11}$$

The example adduced above in Boolean terms has the form:

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{S}^{23}G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The meet of topologies can be described in terms of stretching operators $\mathbf{S}^{pq}$. Indeed, given two topologies associated with the Hasse graphs P and Q, their meet can be obtained by the consequent application of stretching operator $\mathbf{S}$ to P:

$$\mathrm{TCl}\,(P \cup Q) = \mathbf{S}^{pq} \ldots \mathbf{S}^{rs} P$$

where the pairs of indices $pq, \ldots, rs$ run over all arrows of Q, or, equivalently,

$$\mathrm{TCl}\,(P \cup Q) = \mathbf{S}^{ij} \ldots \mathbf{S}^{kl} Q$$

where the pairs $ij, \ldots, kl$ run over all arrows of P.

The operation of strengthening of topology deals with cliques. Recall that a clique Q is a complete subdigraph of G, namely each pair of vertices

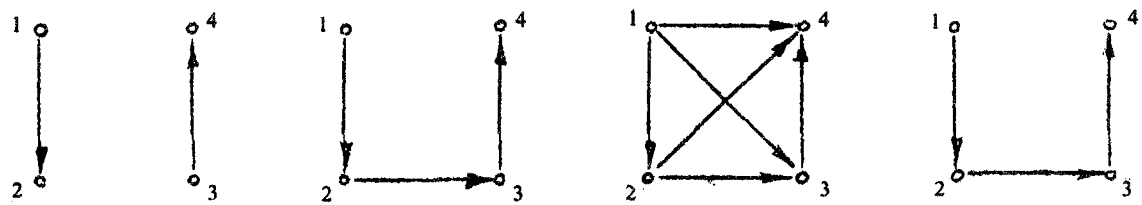

Fig. 5. Strengthening. (a) The graph G, (b) $G \cup \Delta^{23}$, (c) $S^{23}G$, (d) $S^{23}G$, in the brief form with composite arrows omitted.

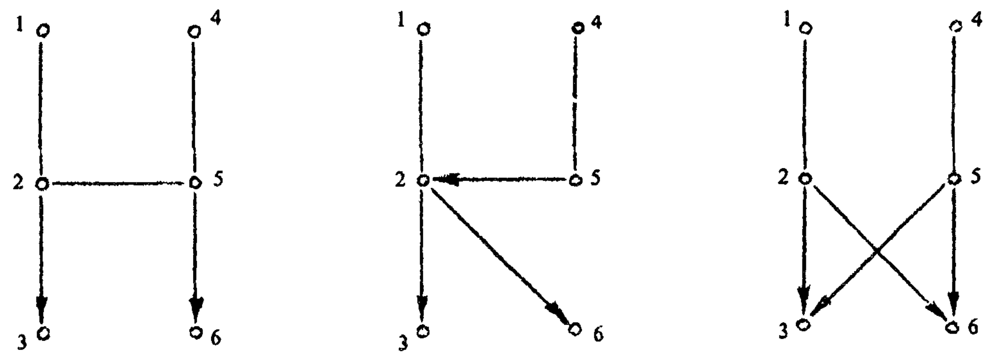

Fig. 6. Cutting.

$q, q' \in Q$ are connected by the arrows $q \to q'$ and $q' \to q$. In particular, each point $x \in X$ is a clique. Considered as components of pattern spaces, *cliques are pointlike objects*. Let P, Q be two nonintersecting cliques. If there is an arrow from a point of P to some point of Q, then due to transitivity, each point of P is connected with each point of Q.

The *cutting operator* $\mathbf{C}^{PQ}$ separates the cliques P and Q by cutting all darts going from P to Q. The special case of cutting operator is dragging out a subclique from a clique (in the case when the union $P \cup Q$ is the clique itself). Consider an example. Let $|X| = 6$ and the topology α be associated with the Hasse graph G. The subsets $P = 12 = \{1, 2\}$ and $Q = 45$ are the cliques. The operator $\mathbf{C}^{PQ}$ drags out P from $P \cup Q = 1245$, while $\mathbf{C}^{QP}\mathbf{C}^{PQ}$ separates P and Q completely (Figure 6).

Remarks on Notation. As I mentioned in Section 2, no ambiguity occurs when some composite arrows are omitted. In addition, the heads on two-sided arrows are omitted. For example, Figure 7 shows graphs that describe the same clique.

To describe this operation in Boolean terms, first introduce the matrix Δ_{ij}^{PQ}:

$$\Delta_{ij}^{PQ} := \begin{cases} 1 & \text{if } i \in P \text{ and } j \in Q \\ 0 & \text{otherwise} \end{cases}$$

Then the cutting operator $\mathbf{C}^{PQ}$ will act as

$$(\mathbf{C}^{PQ}G)_{ij} = \bigvee_{k=1}^{n} (G_{ik}\bar{\Delta}_{ik}^{PQ})(G_{kj}\bar{\Delta}_{kj}^{PQ})$$

So, when a composite arrow is cut, the "wounded" graph "arrows over" due to the transitivity of Hasse graphs.

To clear up the structure of pattern spaces, I introduce one more operator strengthening the topology, called the *discretizer* **DC**, which turns the finite topological space into the discrete union of cliques. In Boolean terms the discretizer has the form

$$(\mathbf{DC}\,G)_{ij} = G_{ij}G_{ji}$$

The action of the discretizer on the graphs considered in the previous example will look as shown in Figure 8. In other words, the discretizer

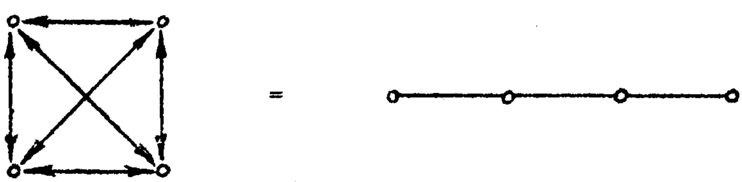

Fig. 7. 4-clique.

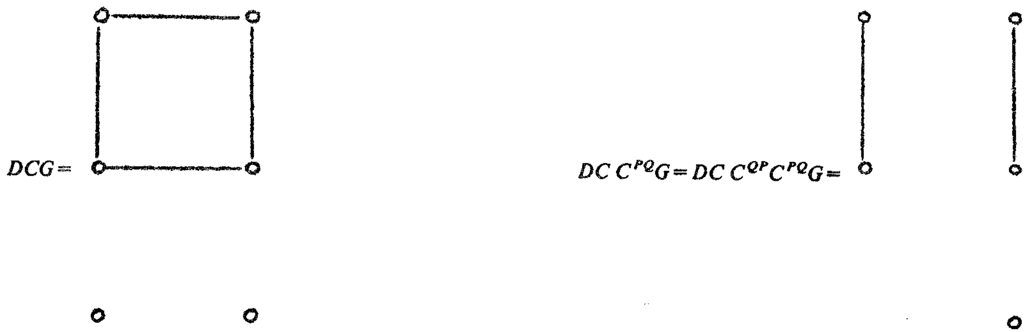

Fig. 8. The discretizer **DC**.

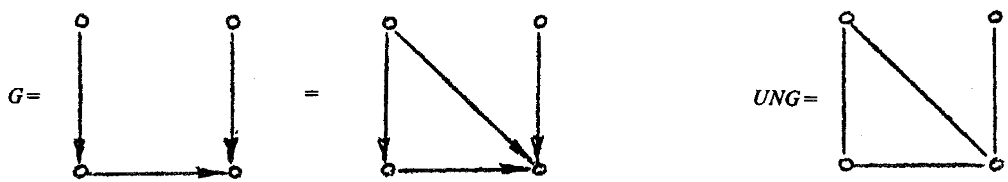

Fig. 9. The underlying graph.

deletes all one-sided arrows. To restore the initial graph G from its discretized remainder **DC** G, the lacking arrows must be added using the stretching operator $\mathbf{S}^{pq}$. Thus, if $H=\mathbf{DC}\,G$, then $G=\mathbf{S}^{pq}\ldots\mathbf{S}^{rs}H$, where the pairs $pq,\ldots,rs$ run over the lacking arrows, while all these lacking arrows are the nonzero elements of the matrix $G_{pq}\bar{G}_{qp}$ (recall that no summation over repeating indices is performed).

Finally, the operator of *underlying graph* construction **UN** adds to each arrow its inverse, making the graph nonoriented (and a non-Hasse graph as well). In Boolean terms the operator **UN** looks like symmetrization:

$$(\mathbf{UN}\,G)_{ij}=G_{ij}\vee G_{ji}$$

An example is given in Figure 9.

4. SURGERY

As already mentioned, the pattern spaces are finite topological (FT) spaces which are patterns of real space (or spacetime) in the sense that they are of the same homotopical type. In this section, I describe the elementary constituents of pattern spaces and the tools for operating with them.

Begin with the study of the connectedness of FT spaces. Let x, y be two points of an FT space X connected by an arrow of the Hasse graph G, say, $x\tau y$. Consider the mapping p: $[0, 1]\to X$ defined as follows:

$$p(t)=\begin{cases}x, & 0\le t\le \frac{1}{2}\\ y, & \frac{1}{2}<t\le 1\end{cases}$$

Let A be an open subset of X and consider its inverse image $p^{-1}(A)$:

$$p^{-1}(A)=\begin{cases}\varnothing & \text{if}\quad \{x, y\}\cap A=\varnothing\\ (\frac{1}{2}, 1] & \text{if}\quad y\in A \text{ and } x\notin A\\ [0, 1] & \text{if}\quad \{x, y\}\subset A\end{cases}$$

Thus $p^{-1}(A)$ is always open in $[0, 1]$. The case $y\notin A$ and $x\in A$ is excluded since A is the open subset of X and contains the monads of each of its elements (see Section 2). That means that there is a continuous mapping p: $[0, 1]\to X$ such that $p(0)=x$ and $p(1)=y$; thus, there is a path from x to y. It is a general topological fact that if there is a path from x to y, then there is always a path from y to x. Therefore, recalling the notion of the underlying graph **UN** G (Section 3), I can assert that if the points x, y are connected by a sequence of arcs of the underlying graph **UN** G, then there is a path from x to y in the topological space X, and vice versa. Due to transitivity of the relation "can be connected by a path," the following two statements are equivalent for any pair x, y of points of X:

(i) There is a path from x to y in the topological space X.

(ii) There is a path from x to y along the underlying graph UN G of the Hasse digraph G associated with the space X.

Now dwell on the elementary constituents of pattern spaces. The first one, which I shall call a cell, is the FT space of four points whose Hasse graph is shown in Figure 10. The cell C is a retractable space, namely each loop in C is homotopic to the constant path. To corroborate this, I adduce the mapping h: $[0, 1] \times [0, 1] \to C$ defined as follows:

$$h(s, t) := \begin{cases} c & \text{if } \frac{1}{3} < s \leq 1 \text{ and } \frac{1}{3} < t < \frac{2}{3} \\ b & \text{if } \frac{2}{3} < s \leq 1 \text{ and } \frac{1}{6} < t \leq \frac{1}{3} \\ d & \text{if } \frac{2}{3} < s \leq 1 \text{ and } \frac{2}{3} \leq t < \frac{5}{6} \\ a & \text{otherwise} \end{cases}$$

which is continuous, and $h(0, t) = a$ and $h(1, t) = abcda$.

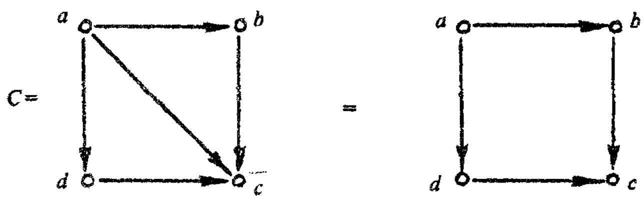

Fig. 10. The cell.

Another elementary constituent is the *hole* H (Figure 11), whose fundamental group is $\mathbf{Z}^1$ since the closed path p: $[0, 1] \to H$,

$$p(t) = \begin{cases} b & \text{if } \frac{1}{6} < t < \frac{1}{3} \\ c & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ d & \text{if } \frac{2}{3} < t < \frac{5}{6} \\ a & \text{otherwise} \end{cases}$$

is unretractable to the constant path $p(t) = a$. It will be shown below that the holes can be obtained by pasting and cutting of a number of cells.

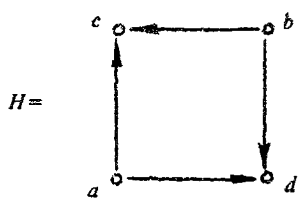

Fig. 11. The elementary hole.

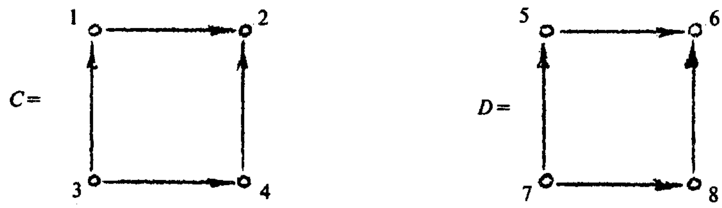

Fig. 12. Two cells before coupling.

The surgeon creates new pattern spaces by sawing (using the stretching operator S) and cutting (using the operator C). As a first example of sawing, consider the coupling of cells. Let C and D be two cells (Figure 12). The coupling will be performed as the following sequence of operations:

$$P_0 = C \cup D, \qquad P_1 = \mathbf{S}^{25} P_0, \qquad P_2 = \mathbf{S}^{52} P_1, \\ P_3 = \mathbf{S}^{47} P_2, \qquad P_4 = \mathbf{S}^{74} P_3 = P_5 \tag{4.1}$$

The transition from P_4 to P_5 is a formal one since the cliques play the role of points in pattern spaces. In other words, the sequence (4.1) of operations pastes the arrows $4 \to 2$ and $7 \to 5$, and I shall call it *pasting*.

Having a storage of cells and pasting them along codirected arrows, the simplest pattern space, a piece of a plane, can be obtained (Figure 14). The composite arrows on the diagram are, as usual, omitted.

There are three kinds of vertices of the Hasse digraph of the plane, for example, a, b, and d. Further pasting yields standard topological patterns: a cylinder, which is obtained by a pasting of the sequence of arrows $(1 \leftarrow 2 \to 3 \leftarrow 4 \to 5)$ with $(6 \leftarrow 7 \to 8 \leftarrow 9 \to 0)$, and a Möbius strip produced

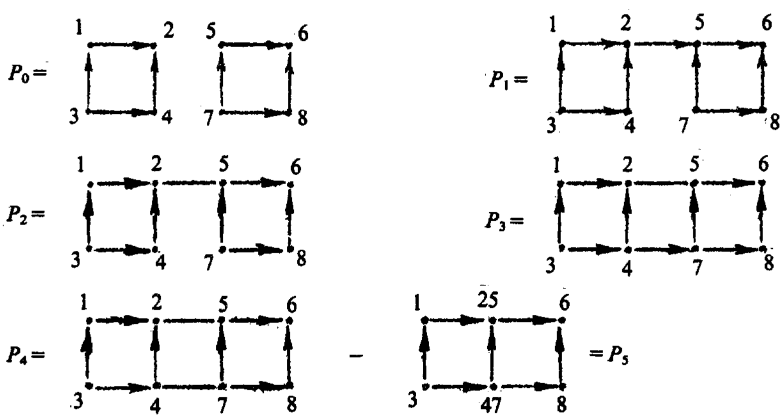

Fig. 13. The stepwise pasting of cells.

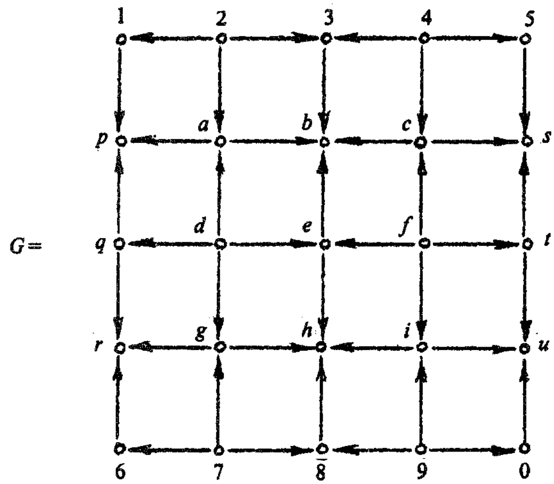

Fig. 14. A piece of plane.

by a pasting of the sequences $(1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5)$ and $(0 \leftarrow 9 \rightarrow 8 \leftarrow 7 \rightarrow 6)$. Their borders are given in Figures 15 and 16, respectively. Pasting the pairs of arrows $(56 \rightarrow 7, 8 \rightarrow 7)$ and $(t \rightarrow u, 01 \rightarrow u)$ by means of the operator $S^{76}S^{78}S^{ut}S^{u0}$ reduces it to the standard hole H (cf. Figures 11 and 16).

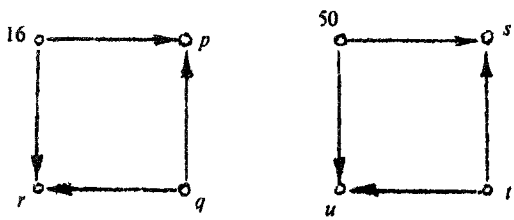

Fig. 15. The border of the cylinder is the pair of holes.

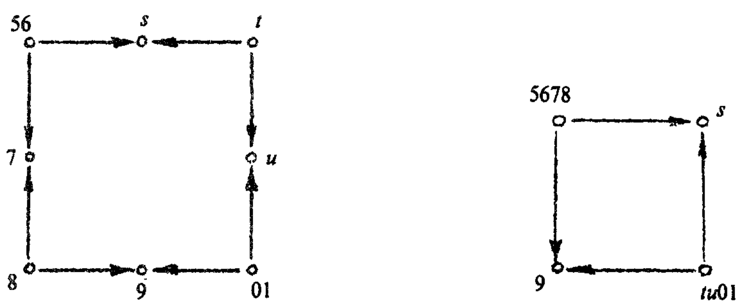

Fig. 16. The border of the Möbius strip: (a) initial, (b) reduced.

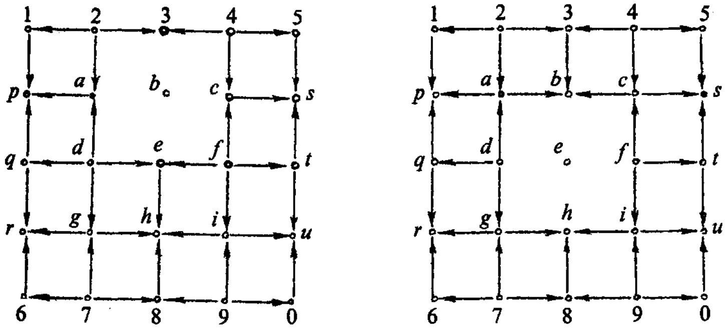

Fig. 17. Making the hole by deleting (a) the point b, by the operator $\mathbf{H}^bG$; (b) the point e, by the operator $\mathbf{H}^eG$.

Another kind of operation performed on the pattern plane (Figure 14) is making a hole. To do this, choose a point, say b, and consequently apply the cutting operators to all its adjacent arrows; as a result, a hole of the kind in Figure 15a appears: $\mathbf{H}^bG=\mathbf{C}^{ba}\mathbf{C}^{bc}\mathbf{C}^{3b}\mathbf{C}^{0b}\mathbf{G}$, while the point b becomes separated (Figure 17a). Deleting the point e from the pattern plane by means of the operator $\mathbf{H}^e=\mathbf{C}^{d0}\mathbf{C}^{f0}\mathbf{C}^{0b}\mathbf{C}^{0h}$ yields the standard hole $dbfh$ (cf. Figures 11 and 17b).

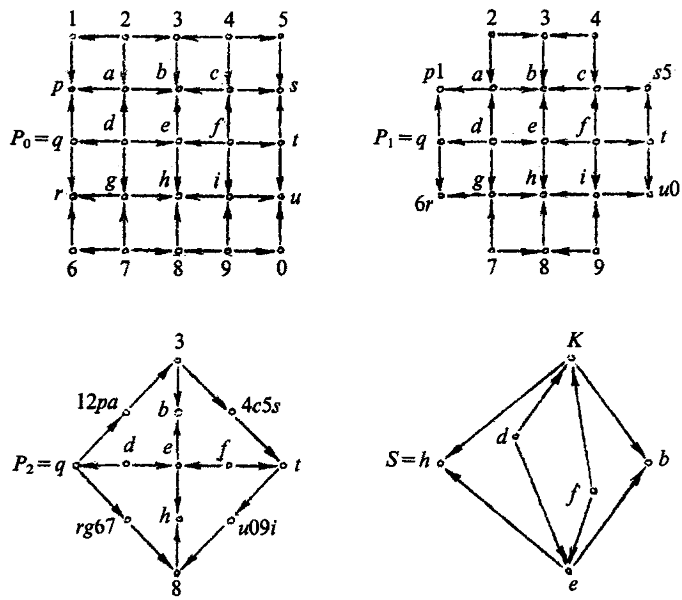

Fig. 18. Stepwise squeezing of the border gives the pattern sphere. (Recall that the points on the figures may denote cliques as well.)

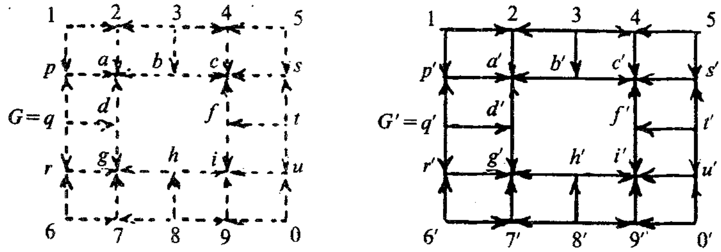

Fig. 19. The holes before pasting.

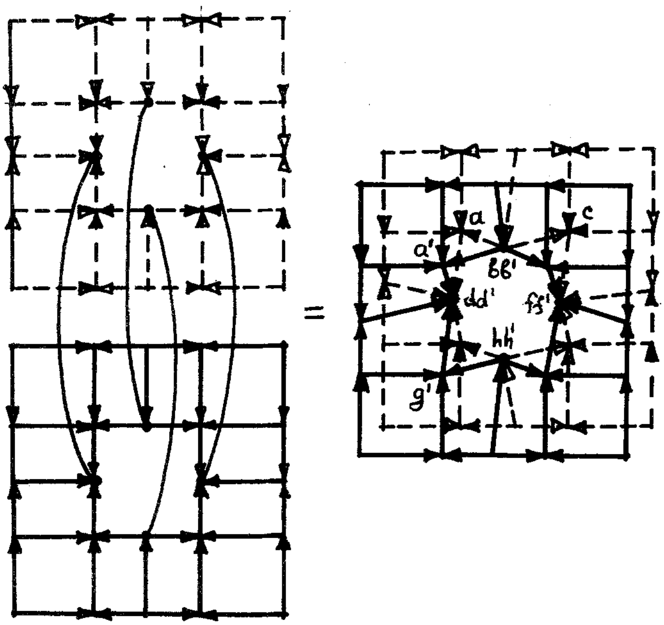

Fig. 20. The pattern wormhole.

The pattern sphere S can be obtained from the pattern plane by squeezing all its border into the clique $K = 12345stu09876rqpacgi$. The squeezing can be performed consecutively (Figure 18): $P_0 = G$ is the pattern plane (Figure 14), $P_1 = \mathbf{S}^{p1}\mathbf{S}^{s5}\mathbf{S}^{u0}\mathbf{S}^{r6}P_0$, $P_2 = \mathbf{S}^{12}\mathbf{S}^{54}\mathbf{S}^{09}\mathbf{S}^{67}P_1$, and $S = \mathbf{S}^{3q}\mathbf{S}^{3t}\mathbf{S}^{8q}\mathbf{S}^{8t}P_2$ (Figure 18).

More complicated patterns may be obtained by making holes and pasting them to the borders of cylinders or other patterns. One more example seems rather attractive. That is the production of a *wormhole*. Given two pattern planes G, G', they can be considered as fragments of a greater pattern space. As a first step (Figure 19), the operators $\mathbf{H}^e$ and $\mathbf{H}^{e'}$ (Figure 17) are applied to G and G'. At a second step (Figure 20), the borders of the holes obtained at the first step are pasted.

Note that the pattern wormhole is rather uniform: it consists of four usual cells $(a \leftarrow bb' \rightarrow a' \rightarrow dd')$, $(c \leftarrow bb' \rightarrow c' \rightarrow ff')$, $(i \leftarrow hh' \rightarrow i' \rightarrow ff')$, and $(g \leftarrow hh' \rightarrow g' \rightarrow dd')$.

5. BOOLEAN SUPERSPACE AND DYNAMICS

In terms of the calculus proposed, the superspace is to some extent the storage of all possible pattern spaces. The natural realization of this idea is to consider a finite set X of points (events) of a fixed cardinality $|X| = n$ and use the lattice $\tau(X)$ of all topologies on X as configuration space (Isham, 1989). However, within the bounds of fixed cardinality the variety of possible topological patterns will always be restricted. The Boolean machinery allows one to broaden this approach.

As described in Section 3, any topology on the finite set X, card $X = n$, can be associated with a Boolean matrix (3.3). The size of the matrix is $n \times n$. To describe the cutting/pasting of pattern spaces, some means to describe separated/unseparated pattern spaces are needed. I claim that the pre-Regge calculus already possesses such means. Namely, two separated pattern spaces can be described as *clopen* (closed and open at once) subsets of a greater topological space. This combined pattern space can in turn be described by the Boolean matrix of size $(m+n) \times (m+n)$, where m, n are the cardinalities of the components. These are the preliminary reasonings motivating the following definition:

The Boolean superspace **BS** is the collection of all Boolean matrices of infinite size satisfying the following conditions:

a. Reflexivity: $G_{ii} = 1$ for any $i = 1, 2, \ldots$.
b. Finiteness: $\text{card}\{G_{ij} | i \neq j \text{ and } G_{ij} = 1\} < \infty$.
c. Transitivity: $(G^m)_{ij} = G_{ij}$ for any integer m.

The condition of finiteness means that any pattern space is considered as a finite topological space accompanied with an infinite number of isolated points, which can be attached for building new patterns.

Both stretching and cutting operators can be evidently extended to the whole **BS** since all their nontrivial action is performed on finite sets. Therefore, given a pair of arbitrary pattern spaces P, Q, both can be considered as elements of **BS**. The transition from P to Q can be performed by consequent application of stretching and cutting operators **S** and **C** or, in other words, as a path in **BS**. Given the initial P and final Q pattern spaces, there are infinitely many paths from P to Q in **BS**. To illustrate this, consider the simplest example: two ways of making the transition of the cell into the hole (Figures 21 and 22).

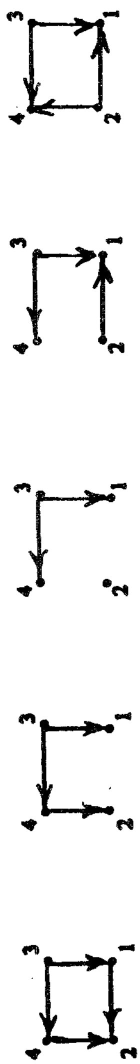

Fig. 21. The path $C^{12}C^{42}S^{21}S^{24}$.

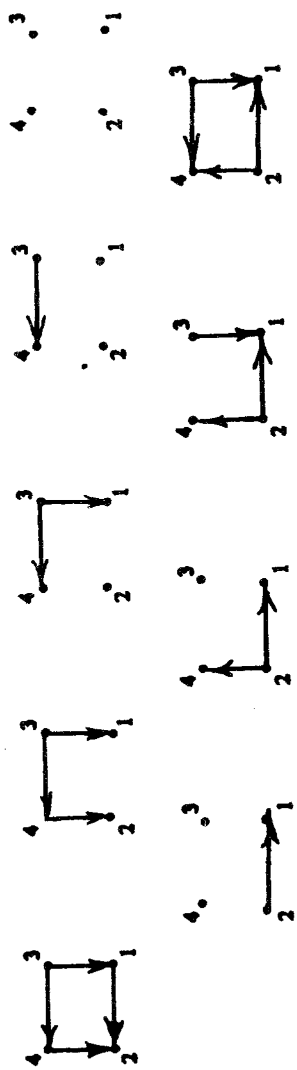

Fig. 22. The path $C^{12}C^{42}C^{31}C^{34}S^{21}S^{24}S^{31}S^{34}$.

The main problem I wish to outline is to *define the amplitudes of elementary transitions*. With the amplitudes defined, each pair of pattern spaces could be associated with the transition amplitude $\langle P|Q\rangle$ defined as a Feynman sum over all possible paths $G_1, \ldots, G_m$ from P to Q:

$$\langle P|Q\rangle = \sum \langle P|G_1\rangle\langle G_1|G_2\rangle \cdots \langle G_m|Q\rangle \tag{5.1}$$

The amplitude of the elementary transitions $\mathbf{S}^{pq}G$ and $\mathbf{C}^{PQ}G$ must have values depending on (i) the form of the Hasse graph G associated with the initial pattern space and (ii) the parameters of the operator **S** or **C**.

6. CONCLUSIONS AND PROSPECTS

First I put together the principal items of the proposed calculus:

1. A finite topological (FT) space, called pattern space, is used to simulate real spacetime up to the homotopical equivalency.

2. Each FT space can be naturally quasiordered by the relation of unseparability (2.3).

3. A finite quasiordered space is associated with the directed Hasse graph whose vertices are points and whose arrows connect quasiordered points.

4. The Hasse graph is associated with its Boolean incidence matrix (3.3).

5. Since all topologies on a given finite set X form a finite lattice, given any pair of topologies on X, the transition between them can be decomposed into a sequence of elementary weakening and strengthening of the topology. This is the way to obtain new topological patterns (Section 4).

6. The pattern superspace is the Boolean space **BS** of all infinite Boolean matrices satisfying certain conditions (Section 5). Thus, given two pattern spaces, the stepwise transition from one of them to another is considered as a *path in* **BS**.

The main problem is to introduce some apparent definition of the amplitude of an elementary transition in order to assign the transition amplitude to any pair of pattern spaces in Feynman's way (5.1); then the quantization of the global topology can be described in terms of pattern spaces.

The variations of pattern space topology can be simulated on a computer by means of Boolean machinery (Section 3). From this point of view, the pattern spaces can be considered as the background for cellular automata whose evolution can be associated with the topological dynamics or motion in the spacetime. I emphasize that in terms of the proposed calculus there is no difference between the spacetime motion and motion in spacetime.

The dynamics of systems described by graphs or Boolean matrices has been studied from various points of view. *Boolean dynamical systems* (Bochmann and Posthoff, 1981) are now mostly used in applied science. However, the Boolean differential calculus worked out within this approach seems to be applicable for pattern space dynamics. The *transition systems* (Finkelstein and Finkelstein, 1982) are abstract automata endowed with a number of controls. In terms of the pre-Regge calculus, these controls might describe elementary transition operators. Finally, the approach based on *simulating automata* (Grib and Zapatrin, 1990, 1991) attempts to introduce transition probabilities without using amplitudes, which are replaced by weights on apparent graphs. It brings nothing new in conventional quantum mechanical computations, but may be a powerful tool in cases when the Hilbert space formalism is not applicable. Note that the lattice of all topologies is complemented but has no unique complements. Thus, it can hardly be embedded into a projector algebra, while the description of such systems in terms of simulating automata does not encounter such difficulties.

ACKNOWLEDGMENTS

I am obliged to Prof. A. Grib, who encouraged me over several years, to my wife Luba for preparation of the manuscript, and to Natalie Perchatkina for the drawings. The hospitality of the Department of Physics of the University of Lecce (Italy) made the timely completion of the paper possible.

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